The (quantum) moment problem: deriving bounds on (quantum) correlations

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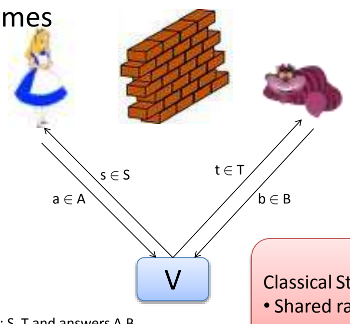
- Wim van Dam, Greg Ver Steeg (In preparation)

Outline

- Games
  - What is known?
  - What we will answer here
- Tools
  - Tensor product vs. commutation relations
  - Positivstellensatz
- Finding bounds using SDPs
  - The idea
  - A hierarchy of SDPs
  - An example
- Generalizations

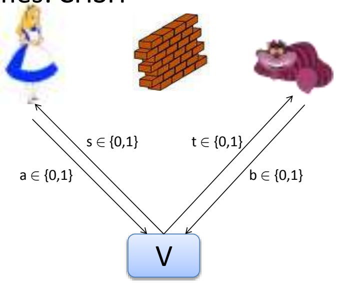
Games



Classical Strategy:
- Shared randomness

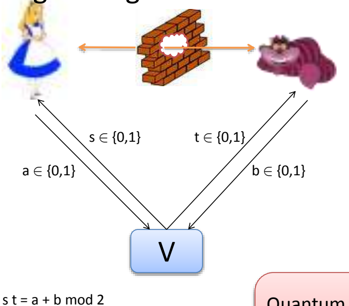
- Sets of questions: S, T and answers A,B
- Probability distribution: π: S $\times$ T $\to$ [0,1]
- Predicate V: A $\times$ B $\times$ S $\times$ T $\to$ {0,1}
  - V(a,b|s,t) = 1 ⇔ Alice and Bob win with answers a and b given questions s and t
- Winning probability
  - ω(G) = max $\Sigma_{s,t}$ π(s,t) $\Sigma_{a,b}$ V(a,b|s,t) Pr[answer a,b| questions s,t]

Games: CHSH



- V(a,b|s,t) = 1 ⇔ s t = a + b mod 2
  ⇔ Alice and Bob win iff s t = a + b mod 2
- Classical value of the game
  - ω(CHSH) = 3/4

Adding entanglement



- V(a,b|s,t) = 1 ⇔ s t = a + b mod 2
  ⇔ Alice and Bob win iff s t = a + b mod 2
- Classical value of the game
  - ω(CHSH) =1/2 + 1/(2 √2) ≈ 0.853

Quantum Strategy:
- Shared state
- Measurements

Goal

- Given a description of a game (S,T,A,B,π,V), what is the maximum probability that the players win the game?
- Compute
  ω(G) = max $\Sigma_{s,t}$ π(s,t) $\Sigma_{a,b}$ V(a,b|s,t) Pr[answer a,b| s,t]

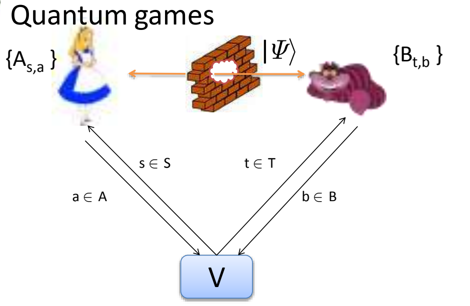
What is known?

- 2-player XOR games
  - Answers are single bits A = B = {0,1}
  - From answers $a \in A$, $b \in B$, the verifier bases his decision only on the XOR c = a + b mod 2
  - Can compute the optimal value in time polynomial in the number of questions using an SDP (Wehner, PRA '06)
- General games
  - Can obtain upper bounds on the optimal value of any game (Navascues, Pironio, Acin, PRL '07)
  - Not efficient
  - Not known whether the optimal value can be reached

What is known?

- 2-prover games
  - Can approximate the value of a unique game to within a certain accuracy in polynomial time (Kempe, Regev, Toner, arxiv:0710.0655)
- General games
  - Here: can find the optimum value of any multi-player game using a hierarchy of SDPs that **converges** to the optimal value of the game. (arxiv:0803.4373, CCC'08)
  - Dual to the new work of Navascues, Pironio, Acin (arxiv:0803.4290).

Quantum games



$\omega(G) = \max \sum_{s,t} \pi(s,t) \sum_{a,b} V(a,b|s,t) \langle\Psi| A_{s,a} \quad B_{t,b} |\Psi\rangle$

Constraints:
- $|| |\Psi\rangle || = 1$
- $A_{s,a} \geq 0$, $B_{t,b} \geq 0$
- $\sum_a A_{s,a} = I$, $\sum_b B_{t,b} = I$
- Tensor product form!

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Tensor product vs. commutation relations

$\omega_f(G) = \sup || \sum_{s,t} \pi(s,t) \sum_{a,b} V(a,b|s,t) A_{s,a} \quad B_{t,b} ||$

- Replacing the tensor product constraint: for all s, a and b,t $[A_{s,a}, B_{t,b}] = 0$
- Gives an upper bound $\omega_f(G) \geq \omega(G)$
- How tight is this upper bound?

Tensor product vs. commutation relations

- By convexity, can assume that the for the optimal solution $A_{s,a}$ (and $B_{t,b}$) cannot be `decomposed'.
- In finite-dimensions, the following two statements are equivalent:
  - For all s,a and t,b, $[A_{s,a}, B_{t,b}] = 0$
  - There exists a partitioning $H = H_A \quad H_B$ such that for all s,a $A_{s,a} \in B(H_A)$ and for all t,b $B_{t,b} \in B(H_B)$
- If the game is such that the optimum value can be achieved in finite dimensions, then $\omega_f(G) = \omega(G)$

The Positivstellensatz

- Consider $q_\nu = \nu I - \sum_{s,t} \pi(s,t) \sum_{a,b} V(a,b|s,t) A_{s,a} B_{t,b}$
- If q_ν is a sum of squares ($q_\nu = \sum_j r_j^\dagger r_j$), then $q_\nu \geq 0$
- But how about the converse?
- ...(almost) given by the Positivstellensatz!
- Variables: $A_{s,a} = A'^\dagger_{s,a} A'_{s,a}$ and $B_{t,b} = B'^\dagger_{t,b} B'_{t,b}$
- Set of polynomials P in such variables.
- Positivity domain
  - $D_P = \{A_{s,a}, B_{t,b} \mid \text{for } p \in P \text{ we have } p(A_{s,a}, B_{t,b}) \geq 0\}$

The Positivstellensatz

- Variables: $A_{s,a} = A'^\dagger_{s,a} A'_{s,a}$ and $B_{t,b} = B'^\dagger_{t,b} B'_{t,b}$
- Set of polynomials P in such variables.
- Positivity domain
  - $D_P = \{A_{s,a}, B_{t,b} \mid \text{for } p \in P \text{ we have } p(A_{s,a}, B_{t,b}) \geq 0\}$
- Helton&McCullough '03 (adapted to complex case):

Let $q_\nu = \nu I - \sum_{s,t} \pi(s,t) \sum_{a,b} V(a,b|s,t) A_{s,a} B_{t,b}$.
If $q_\nu > 0$ for variables in D_P, then there exist polynomials r_j and s_{ij} such that q_ν is a weighted sum of squares:

$$q_\nu = \sum_j r_j^\dagger r_j + \sum_{ij} s_{ij}^\dagger p_i s_{ij}$$

where $p_i \in P$

So what?

- Use the set P to represent the constraints:
  - Commutativity: $i[A_{s,a}, B_{t,b}] = 0$
  - Measurements: $\sum_a A_{s,a} - I = 0$ and $(A_{s,a})^2 - A_{s,a} = 0$
  - Orthogonality: for $a \neq a'$ $i[A_{s,a}, A_{s,a'}] = 0$ and $\{A_{s,a}, A_{s,a'}\} = 0$
  - Let $P = \{\pm i[A_{s,a}, B_{t,b}], \pm(\sum_a A_{s,a} - I), \pm((A_{s,a})^2 - A_{s,a}), \pm i[A_{s,a}, A_{s,a'}], \pm\{A_{s,a}, A_{s,a'}\}, \ldots\}$
- operators $A_{s,b}, B_{t,b} \in D_P$, then they satisfy the constraints

The Positivstellensatz applied

Let $q_\nu = \nu I - \sum_{s,t} \pi(s,t) \sum_{a,b} V(a,b|s,t) A_{s,a} B_{t,b}$.

If $q_\nu > 0$ for variables satisfying our measurement constraints, then there exist polynomials r_j and s_{ij} such that q_ν is a weighted sum of squares:

$$q_\nu = \sum_j r_j^\dagger r_j + \sum_{ij} s_{ij}^\dagger p_i s_{ij}$$

where p_i is a constraint.

- If $q_\nu > 0$, then $\nu I > \sum_{s,t} \pi(s,t) \sum_{a,b} V(a,b|s,t) A_{s,a} B_{t,b}$
- If $q_\nu > 0$ there exists no quantum strategy that attains ν

If there exists no strategy that attains q_ν, then there exist polynomials r_j and s_{ij} such that q_ν is a weighted sum of squares:

$$q_\nu = \sum_j r_j^\dagger r_j + \sum_{ij} s_{ij}^\dagger p_i s_{ij}$$

where p_i is a constraint.

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The idea

- Consider $q_\nu = \nu I - \sum_{s,t} \pi(s,t) \sum_{a,b} V(a,b|s,t) A_{s,a} B_{t,b}$
- If $q_\nu \geq 0$, then $\nu \geq \omega_f(G)$
- Goal: Minimize ν
  - Such that $q_\nu \geq 0$, and the constraints are satisfied.
- If q_ν is a SOS, then $q_\nu \geq 0$
- If $q_\nu > 0$, then
  - q_ν is a weighted SOS $q_\nu = \sum_j r_j^\dagger r_j + \sum_{ij} s_{ij}^\dagger p_i s_{ij}$
  - a SOS for operators satisfying the constraints
- Goal recast: Minimize ν
  - Such that q_ν is a weighted SOS

The idea

- Goal recast: Minimize ν
  - Such that q_ν is a weighted SOS $q_\nu = \sum_j r_j^\dagger r_j + \sum_{ij} s_{ij}^\dagger p_i s_{ij}$
- To cast as an SDP: Minimize ν
  - Such that $q_\nu - \sum_{ij} s_{ij}^\dagger p_i s_{ij}$ is a SOS
- Can be solved using an SDP (analogous Parrillo '00)
  - Fix the total degree of the polynomials
  - Can check whether we find a SOS decomposition for a fixed degree

A hierarchy of SDPs

- To cast as an SDP: Minimize ν
  - Such that $q_\nu - \sum_{ij} s_{ij}^\dagger p_i s_{ij} = \sum_j r_j^\dagger r_j$
- At level n:
  - Fix the degree of r_j to be n and s_{ij} to be n-1
  - Note that the total degree of p_i is at most 2, hence the total degree of q_ν is 2n
- Let ω_n(G) be the optimal value obtained at level n
- If q_ν has a decomposition with degree n, then also with degree n+1
  - $\omega_n(G) \geq \omega_{n+1}(G)$

Reaching the optimal value

Claim: $\lim_{n\to\infty} \omega_n(G) = \omega_f(G)$

- First, note that $\omega_n(G) \geq \omega_f(G)$.
- Second, for $\nu = \omega_f(G) + \epsilon$ with $\epsilon > 0$ SOS

  $q_\nu = \sum_j r_j^\dagger r_j + \sum_{ij} s_{ij}^\dagger p_i s_{ij}$
- Let 2D be the maximum degree of the polynomials.
- Hence at level D we have $\omega_D(G) \leq \omega_f(G) + \epsilon$
- Let $\epsilon \to 0$, can get arbitrarily close

An example: CHSH

- Bell operators $B_{CHSH} = A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2$
- Constraints
  - $(A_1)^2 = (A_2)^2 = (B_1)^2 = (B_2)^2 = I$
  - $[A_1, B_1] = [A_1, B_2] = [A_2, B_1] = [A_2, B_2] = 0$
- $q_\nu = \nu I - B_{CHSH}$
- Goal: minimize ν such that $q_\nu - \sum_{ij} s_{ij}^\dagger p_i s_{ij}$ is SOS
- Let $z = (A_1, A_2, B_1, B_2)$
- Look for Γ such that
  - $z^\dagger \Gamma z = q_\nu - \sum_{ij} s_{ij}^\dagger p_i s_{ij}$
  - $\Gamma \geq 0$

An example: CHSH

- $q_\nu = \nu I - (A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2)$
- Goal: minimize ν such that $q_\nu - \sum_{ij} s_{ij}^\dagger p_i s_{ij}$ is SOS
- Let $z = (A_1, A_2, B_1, B_2)$, look for Γ such that
  - $z^\dagger \Gamma z = q_\nu - \sum_{ij} s_{ij}^\dagger p_i s_{ij}$
  - $\Gamma \geq 0$
  - If $q_\nu - \sum_{ij} s_{ij}^\dagger p_i s_{ij} = \sum_j r_j^\dagger r_j$ then we can find such a Γ
  - For such a Γ, $\Gamma = U^\dagger D U$ and $q_\nu = \sum_j d_j (Uz)_j^\dagger (Uz)_j$

An example: CHSH

- $q_\nu = \nu I - (A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2)$
- Goal: minimize ν such that $q_\nu - \sum_{ij} s_{ij}^\dagger p_i s_{ij}$ is SOS
- Let $z = (A_1, A_2, B_1, B_2)$
- Look for Γ such that

$$z^\dagger \Gamma z = \nu I - (A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2) - \sum_j \gamma_{jj} (I - (A_j)^2) - \sum_j \gamma_{jj} (I - (B_j)^2)$$

$$2\Gamma = \begin{pmatrix} 2\gamma_{11} & 0 & -1 & -1 \\ 0 & 2\gamma_{22} & -1 & 1 \\ -1 & -1 & 2\gamma_{33} & 0 \\ -1 & 1 & 0 & 2\gamma_{44} \end{pmatrix}$$

with $\nu = \gamma_{11} + \gamma_{22} + \gamma_{33} + \gamma_{44}$

An example: CHSH

$$2\Gamma = \begin{bmatrix} 2\gamma_{11} & 0 & -1 & -1 \\ 0 & 2\gamma_{22} & -1 & 1 \\ -1 & -1 & 2\gamma_{33} & 0 \\ -1 & 1 & 0 & 2\gamma_{44} \end{bmatrix}$$

with $\nu = \gamma_{11} + \gamma_{22} + \gamma_{33} + \gamma_{44}$

Minimize $\nu = \mathrm{Tr}(\Gamma)$

Such that $\Gamma \geq 0$

By solving this SDP, one gets $\gamma_{11} = \gamma_{22} = \gamma_{33} = \gamma_{44} = 1/\sqrt{2}$

Giving $\nu = 2\sqrt{2}$

$q_{2\sqrt{2}} = 2\sqrt{2}\, I - B_{CHSH} - 1/\sqrt{2}\, \sum_j (I-(A_j)^2) - 1/\sqrt{2}\, \sum_j (I-(B_j)^2)$
$= z^\dagger \Gamma z = 1/(2\sqrt{2})\, (h_1^\dagger h_1 + h_2^\dagger h_2)$
with $h_1 = A_1 + A_2 - \sqrt{2}\, B_1$ and $h_2 = A_1 - A_2 - \sqrt{2}\, B_2$

Further examples

- Yao's inequality
  - 3 players
  - 3 measurements, 2 outcomes
- I3322 inequality
  - 2 players
  - 3 measurements, 2 outcomes

General idea

- Let z be the vector of monomials up to degree n
- Find Γ such that $z^\dagger \Gamma z = \nu I - B - \sum_{ij} s_{ij}^\dagger p_i s_{ij}$
- Piecewise
  - Choose F_0 such that $z^\dagger F_0 z = \nu I$
  - Choose F_1 such that $z^\dagger F_1 z = - B$
  - Implement constraints individually
    - Write $s_{ij} = \sum_k \alpha_{ijk} z_k$ and note $p_i = \sum_\kappa \beta_{ik} z_k$
    - Can find k' such that $z_{k''} = z_{k'} z_k$
    - Find matrix $F_{ijkk'}$ such that $z^\dagger F_{ijkk'} z = z_{k'} p_i z_{k''} = \sum_\kappa \beta_{ik} z_{k''} z_{k'''}$
  - Minimize ν
  - Such that $F_0 + F_1 + \sum F_{...} \geq 0$

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The quantum moment problem

- Given measurements $\{A_{s,a}\}$ and $\{B_{t,b}\}$
  - Satisfying polynomials (in)equalities
  - With associated probabilities p(a,b|s,t)
- ..does there exist a state ρ_{AB} and measurements such that
  - $P(a,b|s,t) = \mathrm{Tr}(\rho_{AB}\, (A_{s,a} B_{t,b}))$
  - for operators satisfying the (in)equalities
- Can find a certificate that such a state and measurement operators do *not* exist using the Positivstellensatz
- *Different from 'marginal problem': There is no restriction on the dimension*

Generalized theories

- Can do this for any theory for which there exists a corresponding 'Positivstellensatz'
- For example, for a theory described by probabilities p(a,b|s,t), with additional constraints (eg GNST):
  - No-signaling, normalization, positivity…
  - For any state ρ: $\sum_j |E[M_j(\rho)]|^p \leq 1$
  - Note: Probabilities p(a,b|s,t) are real commutative variables
  - Note: Constraints can be expressed by polynomials
- Using the real Positivstellensatz(Parrillo '00)
  - Find certificates certifying that a configuration can't be obtained.
  - Optimize

Motivating the constraint

For Hermitian $\Gamma_1,..\Gamma_N$ satisfying

- $(\Gamma_j)^2 = I$
- for $i \neq j$: $\{\Gamma_i, \Gamma_j\} = 0$

- For any state ρ: $\sum_j \mathrm{Tr}(\Gamma_j \rho)^2 \leq 1$ (Wehner, Winter '07)
- Leads to entropic uncertainty relations that are:
  - Maximally strong for the Shannon entropy
  - Nearly maximally strong for the collision and min-entropy

Motivating the constraint

For Hermitian $\Gamma_1,..\Gamma_N$ satisfying

- $(\Gamma_j)^2 = I$
- for $i \neq j$: $\{\Gamma_i, \Gamma_j\} = 0$

- For any p-state ρ: $\sum_j |\mathrm{Tr}(\Gamma_j \rho)|^p \leq 1$ (van Dam, Ver Steeg, Wehner, '08)
  - allow measurements of $\Gamma_1,..,\Gamma_N$
  - all operations preserving this condition (e.g. Clifford gates)
- Properties
  - Weaker uncertainty relations
  - Stronger non-locality (can have non-local box states for $p=\infty$)
  - ...

Summary and open questions

- Can compute the optimal value of a general multi-party quantum-game, but...
  - Rate of convergence?
  - How hard does it really have to be?
  - Can we say more about tensor products vs. commutation?
- Uncertainty relations for more outcomes?
- What would relaxations of such relations lead to?